Mathematicians' Corner Archive

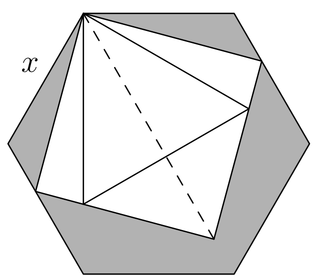
Nathan Kuo

2020-2021

1 A Very Regular Problem (October 5, 2020)

A regular triangle is inscribed in a regular quadrilateral in such a way that the two share a vertex. The quadrilateral is then placed inside a regular hexagon so that all three polygons share a vertex and three of the quadrilateral's vertices lie on the hexagon (including the shared vertex). If a side of the triangle has length 1, the area of the hexagon outside of the quadrilateral can be expressed in the form $\frac{a+\sqrt{b}}{c}$, where a, b, and c are natural numbers and b has no perfect square factors. Find $a+b+c$.

Solution: Recall that a regular polygon has all sides of equal length and all angles of equal measure. Using this, draw a diagram of the problem with an equilateral triangle, a square, and a regular hexagon arranged as required.



Note that by drawing in a diagonal of the square as demonstrated by the dashed line, it can be seen that each half of the square is a 45-45-90 triangle (therefore the ratio of the side length of the square to the diagonal length is $1:\sqrt{2}$) and the diagonal has length $\frac{\sqrt{3}}{2}+\frac{1}{2}$ (30-60-90 triangle and a smaller 45-45-90 triangle), so the square has side length $\frac{1+\sqrt{3}}{2\sqrt{2}}=\frac{\sqrt{6}+\sqrt{2}}{4}$ (or equivalently, $\cos(15°)$). The area of the square is thus

$$\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)^2=\frac{2+\sqrt{3}}{4}.$$

The regular hexagon can be broken down into six congruent equilateral triangles with the same side length as the original hexagon, allowing the area to be calculated using $6\left(\frac{x^2\sqrt{3}}{4}\right)=\frac{3x^2\sqrt{3}}{2}$. Applying the Law of Sines to one of the smaller shaded triangles gives $\frac{\frac{\sqrt{6}+\sqrt{2}}{4}}{\sin(120°)}=\frac{x}{\sin(45°)}$ (notice that all three angles of the triangle, 120°, 15°, and 45°, can be found as a consequence of the regularity of the polygons and symmetry). Solving for x and simplifying gives $\frac{3+\sqrt{3}}{6}$ as the side length of the hexagon. Substituting into the regular hexagon area formula found earlier and simplifying results in

$$\left(\frac{3\sqrt{3}}{2}\right) \cdot \left(\frac{3+\sqrt{3}}{6}\right)^2 = \frac{3+2\sqrt{3}}{4}.$$

Subtracting the area of the square from the area of the hexagon gives

$$\frac{3+2\sqrt{3}}{4} - \frac{2+\sqrt{3}}{4} = \frac{1+\sqrt{3}}{4},$$

so $a+b+c = 1+3+4 = \boxed{8}$.

Congratulations to **Sai Konkimalla**, **Keona Kuo**, **Carol Lu**, and **Stefan Stealey-Euchner** for solving this problem!

2 Not Your Average Snail (October 9, 2020)

Although Speedy the Super Snail can normally only slide at a speed of 1 centimeter (10 millimeters) per second, he, being the ingenious inventor that he is, has created rocket boosters to help other snails in need. Each new booster he activates increases his current speed by $(18b + 30)$ mm/s instantaneously, where b is the number of boosters already in use prior to the one currently being activated. To save his friend Dave from the evil bird Barnabus, he activates a booster each time he travels a distance of 10 centimeters until he reaches Barnabus' nest 33.6 meters away, with his first activation occurring 10 centimeters from his start location. What is his average speed in millimeters per second?

Solution: First, find a recursive formula for Speedy's speed on each constant-length segment of his trip (a relation that outputs his new speed using his previous speed). Indexing each of the 336 10-centimeter long segments from $n = 1$ to $n = 336$, using $s_1 = 10$ mm/s, and the fact that his speed increases by $(18b + 30)$ mm/s between each segment, it can be seen that Speedy's speed follows the formula $s_n = s_{n-1} + 30 + 18(n - 2)$, giving 10 mm/s, 40 mm/s, 88 mm/s, ... Because the sequence has constant second differences, the explicit formula must be quadratic. The general formula of a quadratic function is $ax^2 + bx + c$, and using the first three terms of the sequence and plugging in their respective indices for x, the values of a, b, and c can be determined by solving

$$\begin{cases} a + b + c = 10 \\ 4a + 2b + c = 40 \\ 9a + 3b + c = 88 \end{cases}.$$

This yields $a = 9$, $b = 3$, and $c = -2$, so Speedy's speed s on segment x is given by the function $s(x) = 9x^2 + 3x - 2 = (3x - 1)(3x + 2)$.

Because Speedy travels a constant distance at each speed rather than a constant time, the harmonic mean must be used instead of the arithmetic mean. After converting all distances to millimeters,

$$\frac{33600 \text{ mm}}{\sum_{x=1}^{336} \frac{100 \text{ mm}}{(3x-1)(3x+2) \text{ mm/s}}} = \frac{336}{\sum_{x=1}^{336} \frac{1}{(3x-1)(3x+2)}} = \frac{336}{\frac{1}{2\cdot 5} + \frac{1}{5\cdot 8} + \frac{1}{8\cdot 11} + \cdots + \frac{1}{1007\cdot 1010}}.$$

Notice that the denominator of the fraction is a telescoping series and can be collapsed using partial fractions:

$$\frac{336}{\frac{1}{3}\left(\frac{5-2}{2\cdot 5} + \frac{8-5}{5\cdot 8} + \frac{11-8}{8\cdot 11} + \cdots + \frac{1010-1007}{1007\cdot 1010}\right)} = \frac{336}{\frac{1}{3}\left(\frac{1}{2} - \frac{1}{5} + \frac{1}{5} - \frac{1}{8} + \cdots + \frac{1}{1007} - \frac{1}{1010}\right)}$$

$$= \frac{336}{\frac{1}{3}\left(\frac{1}{2} - \frac{1}{1010}\right)}.$$

Evaluating the result gives $\frac{336\cdot 505}{84} = \boxed{2020}$.

Congratulations to **Stefan Stealey-Euchner**, **Sai Konkimalla**, and **Keona Kuo** for solving this problem!

3 Do the Monster Math! (October 27, 2020)

Frobenia is hoping to buy a cheap house in the infinitely large Number Neighborhood, where the greater the house number, the lower the cost of the house. Number Neighborhood is just like any other neighborhood on every day of the year except for Halloween, when math monsters coordinate attacks on certain house numbers. These monsters can form infinitely many carpools to reach their targeted houses, changing drivers as necessary to attack as many houses as possible but only ever driving forward (increasing house lot numbers). Zombinomials can only drive a distance of exactly 50 lots, vam-pi-res can only drive exactly 79 lots, and diophantoms exactly 85 lots. What is the address of the cheapest house that Frobenia can buy that is guaranteed to be safe from all monster attacks?

Solution: Pending.

4 Candy-natorics (October 28, 2020)

On Halloween, Chloe went trick-or-treating in her neighborhood. She loves M&M's, so she wanted to get as many M&M's as she could this Halloween. At the first house, she was allowed to pick two pieces of candy. If M&M's packets came in two sizes (one with 10 M&M's and one with 15 M&M's), and the distribution of candy in the bowl was as follows:

- 15 KitKats,
- 15 large M&M's packets,
- 20 small M&M's packets,
- 20 Starbursts,
- 30 Hershey's bars,

what was the expected number of M&Ms she could get from this house if she picked the candy randomly? Write your answer in the form $\frac{m}{n}$, where m and n are relatively prime, positive integers.

Solution: The main idea behind solving this question is to break the problem into smaller cases which are easier to deal with. We will break the problem into cases based on the number of M&M's she picks:

Case 1: Chloe picks two large M&M's packets (a total of 30 M&M's).

There are a total of $\binom{15}{2}$ ways to pick two large M&M's packets.

Case 2: Chloe picks one small M&M's packet and one large M&M's packet (a total of 25 M&M's).

There are a total of $15 \cdot 20$ ways to pick the two M&M's packets.

Case 3: Chloe picks two small M&M's packets (a total of 20 M&M's).

There are a total of $\binom{20}{2}$ ways to pick two small M&M's packets.

Case 4: Chloe picks only one large M&M's packet (for a total of 15 M&M's). She can pick the large M&M's packet in 15 ways, and she pick another piece of candy in 65 ways for a total of $15 \cdot 65$ ways to pick two pieces of candy.

Case 5: Chloe picks only one small M&M's packet (for a total of 10 M&M's).

She can pick the small M&M's packet in 20 ways, and she can pick the other piece of candy in 65 ways for a total of $20 \cdot 65$ ways.

Case 6: Chloe picks no M&M's packets (for a total of 0 M&M's).

She can pick the candy in $\binom{65}{2}$ ways.

Now that we have our different cases, we can calculate the expected number of M&M's she will get. We calculate this by multiplying the number of M&M's in each case by the total number of ways to achieve that case. Also, we have to divide by the total number of ways to pick two candies because we are looking for expected value. The total number of possible candy combinations which Chloe can pick is $\binom{100}{2}$.

In total, we get that the expected value is the following:

$$\text{M\&M's} = \frac{30\binom{15}{2} + 25(15 \cdot 20) + 20\binom{20}{2} + 15(15 \cdot 65) + 10(20 \cdot 65) + 0\binom{65}{2}}{\binom{100}{2}}.$$

Simplifying the above expression, we get that the expected number of M&M's is the following:

$$\text{Expected Number of M\&M's} = \boxed{\frac{17}{2}} \text{ M\&M's}$$

Congratulations to **Stefan Stealey-Euchner**, **Carol Lu**, **Keona Kuo** for solving this problem!

Credit to **Sai Konkimalla** for this problem and solution.