Counting and Probability Solutions

Oshadha Gunasekara

August 30, 2015

1 Solutions

1. **Problem:** How many terms are in the following sequence: $5, 8, 11, \ldots, 302$?

   Solution: We notice that the sequence follows a pattern: start with 5 and subtract 3 from the previous term to get the next term. Therefore, we can easily transform the sequence to a consecutive sequence beginning with 1 by subtracting 2 from each term, then dividing each term by 3.

   $$5, 8, 11, \ldots, 302 \rightarrow 3, 6, 9, \ldots, 300 \rightarrow 1, 2, 3, \ldots, 100$$

   Therefore, there are **100** terms in the sequence.

2. **Problem:** John needs 3 strong people to help him move his furniture into his new house. He remembers that he has 11 friends who constantly talk about the gym. How many ways can he choose 3 people from his 11 strong friends?

   Solution: Because the problem does not require that John choose his friends in different orders, order does not matter and we must use combinations. He is choosing 3 from a group of 11. Therefore,

   $$\binom{11}{3} = \mathbf{165} \text{ ways.}$$

3. **Problem:** How many ways can 12 be written as the sum of 4 positive digits?

   Solution: This problem requires casework. It is important to be organized when dealing with casework because it can easily get out of hand. The following shows one way of representing the cases.

   $(1, 1, 1, 9)$ $(1, 1, 2, 8)$ $(1, 1, 3, 7)$ $(1, 1, 4, 6)$ $(1, 1, 5, 5)$
   $(1, 2, 2, 7)$ $(1, 2, 3, 6)$ $(1, 2, 4, 5)$
   $(1, 3, 3, 5)$ $(1, 3, 4, 4)$

   $(2, 2, 2, 6)$ $(2, 2, 3, 5)$ $(2, 2, 4, 4)$
   $(2, 3, 3, 4)$

   $(3, 3, 3, 3)$

   This gives us **15** different ways.

4. **Problem:** (iTest 2007 #4) Star flips a quarter four times. Find the probability that the quarter lands heads exactly twice.

   Solution: We can represent a Tails being flipped with a T and a Heads being flipped with a H. As such, we can represent flipping exactly two Heads with HHTT. Finding the total number of ways this can be arranged will give us the amount of ways Heads can be flipped exactly twice. The number of arrangements of the group of letters HHTT is $\frac{4!}{2!\cdot 2!} = 6$. There are a total of 2^4 ways a quarter can be flipped four times. Therefore, the probability that two of the flips are heads is $\frac{6}{2^4} = \mathbf{\frac{3}{8}}$.

5. **Problem:** (AMC10 2004A #12) Henry's Hamburger Heaven orders its hamburgers with the following condiments: ketchup, mustard, mayonnaise, tomato, lettuce, pickles, cheese, and onions. A customer can choose one, two, or three meat patties, and any collection of condiments. How many different kinds of hamburgers can be ordered?

   Solution: For each condiment, a customer may either order it or not. There are 8 condiments. Therefore, there are $2^8 = 256$ ways to order the condiments. There are also 3 choices for the meat, making a total of $256 \cdot 3 = \mathbf{768}$ possible hamburgers from the Fundamental Counting Principle.

6. **Problem:** A die is rolled 4 times. What is the probability that at least a 3 is rolled each time?

   Solution: Rolling a die four times constitute four independent events. Since each roll is aiming to get at least a 3, the probability for each event will be equal. There are 4 favorable outcomes: $3, 4, 5, 6$. There are 6 total outcomes: $1, 2, 3, 4, 5, 6$. Thus the probability for one event is $\frac{4}{6} = \frac{2}{3}$. Therefore, the probability of rolling a die four times and getting at least a 3 each time is
   $$(\frac{2}{3})^4 = \mathbf{\frac{16}{81}}$$

7. **Problem:** How many ways can 5 distinguishable balls be placed in 10 indistinguishable boxes?

   Solution: First, we must notice that the question speaks of "indistinguishable boxes." Since the boxes cannot be distinguished, we must find how many ways the distinguishable balls can be ordered. They can be ordered in 7 different ways, shown below.

   (1) 1 1 1 1 1
   (2) 1 1 1 2
   (3) 1 1 3
   (4) 1 2 2
   (5) 1 4
   (6) 2 3
   (7) 5

   Now, we must find the number of ways that distinguishable balls can be placed into each arrangement. (1) and (7) both have 1 way because it does not matter which order they are in. (2) has $\binom{5}{2}$ ways because you must choose 2 balls to go into one box while the rest go into others. (3) has $\binom{5}{3}$ ways because you must choose 3 balls to go into one box. (4) has $\frac{\binom{5}{2}\cdot\binom{3}{2}}{2}$ ways, because we must account for indistinguishability of the boxes. (5) has $\binom{5}{4}$ ways because you must choose 4 balls to go into one box. (6) has $\binom{5}{2}$ ways because you need to choose 2 balls to go into one box. Adding up these values, we find
   $$1 + 1 + 10 + 10 + 15 + 5 + 10 = \mathbf{52} \text{ ways}$$

8. **Problem:** (AMC10 2004B #2) How many two-digit positive integers have at least one 7 as a digit?

   Solution: We can use complementary counting. The complement of having at least one 7 as a digit is having no 7s as a digit. We have 9 digits to choose from for the first digit, since 0 cannot be the first digit, and 10 digits for the second. As such, we have $9 \cdot 10 = 90$ total two-digit numbers. Since we cannot have 7 as a digit, we have 8 first digits and 9 second digits to choose from. Thus, there are $8 \cdot 9 = 72$ two digit numbers without a 7 as a digit. Therefore, there are

$$90 - 72 = \mathbf{18} \text{ integers}$$

9. **Problem:** (AMC10 2001 #19) Pat wants to buy four donuts from an ample supply of three types of donuts: glazed, chocolate, and powdered. How many different selections are possible?

   Solution: Here, we can use stars and bars in a creative way. Let the donuts be represented by stars. We wish to find all possible combinations of glazed, chocolate, and powdered donuts. This gives us 3 groups and 4 donuts. Since there does not have to be a donut of each type, we use 6 stars, 2 for the dividers and 4 for the donuts. We need to choose 2 out of the 6. Therefore, there are

$$\binom{6}{2} = \mathbf{15} \text{ possible selections}$$

10. **Problem:** (AJHSME 1985 #15) How many whole numbers between 100 and 400 contain the digit 2?

    Solution: Here, we can use complementary counting. The total amount of numbers between 100 and 400 can be found by subtracting 100 from 400 and subtracting 1, because the word "between" makes the set exclusive. This gives us a total of 299 numbers. Now we need to find how many numbers do not have the number 2. There are 2 possible digits for the hundred's digit $(1, 3)$, 9 possible digits for the ten's digit $(1, 3, 4, \ldots, 9, 0)$ and 9 possible digits for the one's digit $(1, 3, 4, \ldots, 9, 0)$. This gives us $2 \cdot 9 \cdot 9 = 162$ numbers. However we must subtract 1 because we included 100 in this count, giving us $162 - 1 = 161$ numbers without a 2. Therefore, there are

$$299 - 161 = \mathbf{138} \text{ numbers satisfying the condition}$$

11. **Problem:** (AMC12 2005A #11) How many three-digit numbers satisfy the property that the middle digit is the average of the first and the last digits?

    Solution: Let the digits be A, B, C so that $B = \frac{A+C}{2}$. In order for this to be an integer, A and C have to have the same parity. This means they must be either both even or both odd. There are 9 possibilities for A $(1, 2, 3, \ldots, 8, 9)$ and 5 for C (each digit has 5 other digits which have the same parity). Moreover, there is 1 possible value of B for each pair (A, C). Therefore, there are

$$9 \cdot 5 \cdot 1 = \mathbf{45} \text{ numbers}$$

12. **Problem:** What is the maximum number of possible points of intersection of 100 circles?

    Solution: This is a pattern recognition problem. We must find a simple way to find the maximum number of intersections of n circles. Notate $I(n)$ to be the maximum number of intersections of n circles.

$$I(0) = 0$$
$$I(1) = 0$$
$$I(2) = 2$$
$$I(3) = 6$$
$$I(4) = 12$$

    We begin to notice a pattern. Starting from $I(0)$, we add 0, then 2, then 4, then 6. We are adding consecutive even numbers. Because of this, we can base our general formula on the formula for the sum of n consecutive even numbers, $n(n+1)$. We notice that the sequence seems to be shifted up by one term. Thus, our general formula will be $I(n) = n(n-1)$. From this,

$$I(100) = 100 \cdot 99 = \mathbf{9900} \text{ intersections}$$

13. **Problem:** What is the coefficient of x^3 in the expansion of the following: $(1 + x + x^2 + x^3 + x^4 + x^5)^6$?

    Solution: This problem uses an interesting property of polynomial multiplication. That is, when multiplying two polynomials $(1+x+x^2+x^3+\ldots+x^n)$ and $(1+x+x^2+x^3+\ldots+x^m)$, the coefficient of the x^k term can be found by counting the number of ways k can be made using the sum of an exponent of the first polynomial and an exponent from the second polynomial. For example, if we wish to find the coefficient of x^3 of $(1 + x + x^2)^2$, we find how many ways we can make 3 adding one element from each of the two sets $\{0, 1, 2\}$ and $\{0, 1, 2\}$. We find that there are 2 ways to do this ($(1, 2)$ and $(2, 1)$). Remember, order does matter. Thus the coefficient of x^3 is 2.

    Back to the problem, we see a similar situation. In this case, we have six identical polynomials. Therefore, we can essentially rewrite the question into "How many ways can we write 3 as the sum of 6 non-negative integers, where order does matter?" This is a simple stars and bars application. We have 8 stars, 3 representing the number and 5 representing the dividers. We need to choose 5 dividers. Therefore, we the coefficient of x^3 is

    $$\binom{8}{5} = \mathbf{56}$$

14. **Problem:** (AMC10 2002B #9) Using the letters A, M, O, S, and U, we can form 120 five-letter "words". If these "words" are arranged in alphabetical order, then the "word" $USAMO$ occupies which position?

    Solution: There are $4! \cdot 4$ "words" beginning with each of the first four letter alphabetically. From there, there are $3! \cdot 3$ "words" with U as the first letter and each of the first three letters alphabetically. After that, the next "word" is $USAMO$. Therefore, the answer is

    $$4! \cdot 4 + 3! \cdot 3 + 1 = \mathbf{115}$$

15. **Problem:** (AMC10 2001 #25) How many positive integers not exceeding 2001 are multiples of 3 or 4 but not 5?

    Solution: This problem is a fairly direct application of the Principle of Inclusion and Exclusion. First, we find the numbers not exceeding 2001 that are multiples of 3 or 4. From PIE, this is $\lfloor \frac{2001}{3} \rfloor + \lfloor \frac{2001}{4} \rfloor - \lfloor \frac{2001}{12} \rfloor = 1001$ numbers. To account for the "but not 5" in the problem, we must subtract the numbers not exceeding 2001 that are multiples of 15 (LCM of 3 and 5) or 20 (LCM of 4 and 5). From PIE, this is $\lfloor \frac{2001}{15} \rfloor + \lfloor \frac{2001}{20} \rfloor - \lfloor \frac{2001}{60} \rfloor = 200$ numbers. Therefore, the amount of numbers not exceeding 2001 that are multiples of 3 or 4, but not 5 is

    $$1001 - 200 = \mathbf{801}$$

16. **Problem:** (AMC10 2002A #22) A set of tiles numbered 1 through 100 is modified repeatedly by the following operation: remove all tiles numbered with a perfect square, and renumber the remaining tiles consecutively starting with 1. How many times must the operation be performed to reduce the number of tiles in the set to one?

    Solution: If we repeatedly record the number of tiles left after the set is modified, we can see the number of tiles reduces to the next lowest square after 2 iterations. This can be seen in Table 1 on the next page.

    By this method, we find that the operation must be performed **18** times.

Table 1: Removing Tiles

Number	Removed	Left
1	10	90
2	9	81
3	9	72
4	8	64
5	8	56
6	7	49
7	7	42
...	...	...
16	2	4
17	2	2
18	1	1

17. **Problem:** (AMC10 2003A #21) Pat is to select six cookies from a tray containing only chocolate chip, oatmeal, and peanut butter cookies. There are at least six of each of these three kinds of cookies on the tray. How many different assortments of six cookies can be selected?

    Solution: We must find the amount of ways that we can split the 6 cookies into 3 groups, where each group does not have to have a cookie. Here we have 8 stars, 6 representing the cookies and 2 representing the dividers. We need to choose 2 dividers from the 8 stars. Therefore, there are

    $$\binom{8}{2} = \mathbf{28} \text{ different assortments}$$

18. **Problem:** (AMC10 2006B #17) Bob and Alice each have a bag that contains one ball of each of the colors blue, green, orange, red, and violet. Alice randomly selects one ball from her bag and puts it into Bob's bag. Bob then randomly selects one ball from his bag and puts it into Alice's bag. What is the probability that after this process the contents of the two bags are the same?

    Solution: Since there is the same amount of balls in Alice's bag and Bob's bag, and there is an equal chance of each ball being selected, the color of the ball that Alice puts in Bob's bag doesn't matter. Without loss of generality, let the ball Alice puts in Bob's bag be red. For both bags to have the same contents, Bob must select one of the 2 red balls out of the 6 balls in his bag. Therefore, the desired probability is $\frac{2}{6} = \mathbf{\frac{1}{3}}$.

19. **Problem:** (AIME I 2002 #1) Many states use a sequence of three letters followed by a sequence of three digits as their standard license-plate pattern. Given that each three-letter three-digit arrangement is equally likely, the probability that such a license plate will contain at least one palindrome (a three-letter arrangement or a three-digit arrangement that reads the same left-to-right as it does right-to-left) is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

    Solution: Using complementary counting, we count all of the license plates that do not have the desired property. In order to not be a palindrome, the first and third characters of each string must be different. Therefore, there are $10 \cdot 10 \cdot 9$ three digit non-palindromes, and there are $26 \cdot 26 \cdot 25$ three letter non-palindromes. Since there are $10^3 \cdot 26^3$ total three-letter three-digit arrangements, the probability that a license plate does not have the desired property is

$\frac{10\cdot10\cdot9\cdot26\cdot26\cdot25}{10^3\cdot26^3} = \frac{45}{52}$. We subtract this from 1 to get the desired probability, $\frac{7}{52}$. Therefore, the answer is $7 + 52 = \mathbf{59}$.

20. **Problem:** (AIME I 2012 #1) Find the numbers of positive integers with three not necessarily distinct digits, abc, with $a \neq 0$ and $c \neq 0$ such that both abc and cba are multiples of 4.

    Solution: A positive integer is divisible by 4 if and only if its last two digits are divisible by 4. For any value of b, there are two possible values for a and c, since we find that if b is even, a and c must be either 4 or 8, and if b is odd, a and c must be either 2 or 6. Therefore, there are $2 \cdot 2 = 4$ ways to choose a and c for each b, and 10 ways to choose b since b can be any digit. Therefore, the answer is $4 \cdot 10 = \mathbf{40}$.