Counting and Probability

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1 Introduction

1.1 Counting

Counting forms the basis of many combinatorics problems. In order to delve into more difficult problems, it is essential that we cover the basics. First, we must learn to see patterns and use them to our advantage. Many advanced problems require the solver to use pattern recognition skills to deduce certain parts of the question.

Example: How many terms are in the following sequences:

1. $1, 2, 3, 4, \ldots, 100$
2. $5, 6, 7, 8, \ldots, 134$
3. $2, 4, 6, 8, \ldots, 200$
4. $1, 4, 9, 16, \ldots, 196$
5. $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \ldots, \frac{1}{420}$

1.2 Probability

Questions regarding probability come in a variety of topics, including Algebra, Geometry, Counting, etc. Any two events can be characterized as either *independent* or *dependent*. Independent events are not affected by the outcome of other events. For example, rolling a die twice consists of two independent events. If we wish to find a probability of two independent events occurring simultaneously, we would *multiply* the probabilities of each event happening separately.

Example: Two dice are rolled. What is the probability of getting an even number on the first roll and a multiple of 3 on the second?

Dependent events are affected by the outcome of other events. For example, picking balls out of a hat without replacement consists of dependent events. After picking one ball, the amount of balls left in the hat decreases, altering the probability of choosing the next ball. If we wish to find a probability of two dependent events occurring simultaneously, we would *multiply* the probabilities of each event happening concurrently.

Example: Suppose you have a standard 52 card deck. What is the probability of choosing a Jack, then a King, without replacement?

2 Fundamental Counting Principle

Theorem 1. *If event E_1 has a_1 ways of occurring, event E_2 has a_2 ways of occurring, …, and event E_n has a_n ways of occurring, then there are $a_1 a_2 a_3 \dots a_n$ ways that all of the events $E_1, E_2, \dots, E_n$ occur simultaneously.*

The Fundamental Counting Principle gives us a way to find out the total number of ways that different events can occur. The primary example of this principle is choosing a meal from a menu given some choices.

> **Example:** Ricky decided to go to a local Chipotle to eat a quick dinner. He could either get a burrito or a bowl. There were five meats, two types of rice, two types of beans, and eight toppings. If Ricky wants a burrito or a bowl and one each of meats, rice, beans, and toppings, how many options does Ricky have?

3 Permutations and Combinations

3.1 Introduction

Theorem 2. *A permutation is a way in which a set of objects or numbers can be ordered, or rearranged. It is normally notated as $P(n, k)$, where*

$$P(n,k) = \frac{n!}{(n-k)!}$$

> **Example:** Bob considers himself an autocrat. He needs to choose students to be in the student council. There are 3 positions to fill: Vice-President, Secretary, and Treasurer. He doesn't need to choose a President because he chose himself. If there are 10 viable candidates for student council, how many different ways can the three positions be filled?

Theorem 3. *A combination is a way of selecting a set of objects or numbers out of a larger group, where order does not matter. It is normally notated as $C(n, k)$ or $\binom{n}{k}$ where*

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

> **Example:** Matt recently won the Fish-are-Friends-not-Food raffle that he entered. He won 5 tickets to Hawaii! He is allowed to bring 4 of his friends to Hawaii with him. If Matt has 15 friends, how many ways can he choose 4 of them to bring with him on the trip?

3.2 Stars and Bars

Stars and Bars, also referred to as Balls and Urns, is a popular method used in combinatorics. It is primarily used to solve problems of the form: how many ways can one distribute k indistinguishable objects into n bins? This can be re-interpreted to: how we organize k stars amongst $n-1$ dividers? To solve this problem, we must realize that if we place 1 divider between a group of objects, it divides into a group of 2. More generally, using n dividers splits a group up into $n+1$ groups. Therefore, we must find how many ways we can put the dividers between the objects.

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Example: How many ways can 10 identical manuals for world domination be split into 3 distinguishable containers, where each container has to have at least 1 manual.

Example: How many ways can 10 identical manuals for world domination be split into 3 distinguishable containers, where each container does not have to have at least 1 manual.

3.3 Distinguishibility

Whether or not an object is distinguishable can determine how a problem is solved. Objects are *distinguishable* if they can be distinctly ordered. For example, if there are three balls labeled A, B, and C, one can make a distinction between a line of balls ABC and BCA.

Example: How many ways can three distinguishable balls be placed in 5 distinguishable boxes?

Objects are *indistinguishable* if they cannot be distinctly ordered. For example, if there are three indistinguishable balls in a line, they will appear the same no matter what order they are in.

Example: How many ways can three indistinguishable balls be placed in 5 distinguishable boxes?

4 Complementary Counting

Theorem 4. *Complementary counting is counting the complement of the set we want to count, and subtracting that from the total number of possibilities.*

Counting the complement instead of the actual value asked can really help. It can reduce the time it takes to solve a problem and often times the complexity of the problem as well. A big hint that a question can be simplified by complementary counting is the phrase "at least."

Example: How many ways can 5 coins be flipped such that there is at least 1 heads?

5 Principle of Inclusion and Exclusion

Theorem 5. *The Principle of Inclusion-Exclusion (abbreviated PIE) provides an organized method/formula to find the number of elements in he union of a given group of sets, the size of each set, and the size of all possible intersections among the sets. If we are given the sizes of two sets,* $|A_1|$ *and* $|A_2|$*, and their intersection* $|A_1| \cap |A_2|$*, then their union is:*

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

Example: Sloan has two groups of followers: the cultists and worshipers. He has 43 total followers. He has 27 cultists and 29 worshipers. How many of Sloan's followers are both cultists and worshipers?

6 Problems

1. How many terms are in the following sequence: $5, 8, 11, \ldots, 302$?

2. John needs 3 strong people to help him move his furniture into his new house. He remembers that he has 11 friends who constantly talk about the gym. How many ways can he choose 3 people from his 11 strong friends?

3. How many ways can 12 be written as the sum of 4 positive digits?
4. (iTest 2007 #4) Star flips a quarter four times. Find the probability that the quarter lands heads exactly twice.
5. (AMC10 2004A #12) Henry's Hamburger Heaven orders its hamburgers with the following condiments: ketchup, mustard, mayonnaise, tomato, lettuce, pickles, cheese, and onions. A customer can choose one, two, or three meat patties, and any collection of condiments. How many different kinds of hamburgers can be ordered?
6. A die is rolled 4 times. What is the probability that at least a 3 is rolled each time?
7. How many ways can 5 distinguishable balls be placed in 10 indistinguishable boxes?
8. (AMC10 2004B #2) How many two-digit positive integers have at least one 7 as a digit?
9. (AMC10 2001 #19) Pat wants to buy four donuts from an ample supply of three types of donuts: glazed, chocolate, and powdered. How many different selections are possible?
10. (AJHSME 1985 #15) How many whole numbers between 100 and 400 contain the digit 2?
11. (AMC12 2005A #11) How many three-digit numbers satisfy the property that the middle digit is the average of the first and the last digits?
12. What is the maximum number of possible points of intersection of a 100 circles?
13. What is the coefficient of x^3 in the expansion of the following: $(1 + x + x^2 + x^3 + x^4 + x^5)^6$?
14. (AMC10 2002B #9) Using the letters A, M, O, S, and U, we can form 120 five-letter "words". If these "words" are arranged in alphabetical order, then the "word" $USAMO$ occupies which position?
15. (AMC10 2001 #25) How many positive integers not exceeding 2001 are multiples of 3 or 4 but not 5?
16. (AMC10 2002A #22) A set of tiles numbered 1 through 100 is modified repeatedly by the following operation: remove all tiles numbered with a perfect square, and renumber the remaining tiles consecutively starting with 1. How many times must the operation be performed to reduce the number of tiles in the set to one?
17. (AMC10 2003A #21) Pat is to select six cookies from a tray containing only chocolate chip, oatmeal, and peanut butter cookies. There are at least six of each of these three kinds of cookies on the tray. How many different assortments of six cookies can be selected?
18. (AMC10 2006B #17) Bob and Alice each have a bag that contains one ball of each of the colors blue, green, orange, red, and violet. Alice randomly selects one ball from her bag and puts it into Bob's bag. Bob then randomly selects one ball from his bag and puts it into Alice's bag. What is the probability that after this process the contents of the two bags are the same?
19. (AIME I 2002 #1) Many states use a sequence of three letters followed by a sequence of three digits as their standard license-plate pattern. Given that each three-letter three-digit arrangement is equally likely, the probability that suck a license plate will contain at least one palindrome (a three-letter arrangement or a three-digit arrangement that reads the same left-to-right as it does right-to-left) is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.
20. (AIME I 2012 #1) Find the numbers of positive integers with three not necessarily distinct digits, abc, with $a \neq 0$ and $c \neq 0$ such that both abc and cba are multiples of 4.