Sprint Test Round 11601 Solutions

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1 Solutions

1. **Problem:** If $x + 2y = 4$ and $3x + 5y = 7$, what is the value of $4x + 6y$?

   Solution: We can manipulate the given equations to get the expression we wish for.

   $$\begin{aligned} 3x + 5y &= 7 &(1)\\ -(x + 2y) &= -4 &(2)\\ 2x + 3y &= 3\\ 4x + 6y &= 6 \end{aligned}$$

   Therefore, our answer is 6 **(E)**.

2. **Problem:** What is the area of a square inscribed in a circle with circumference 16π?

   Solution: A circle with circumference 16π has radius 8. If we draw the diagram, we find the area of the square will be $2r^2$. Therefore, the area of the square will be $A = 2 \cdot 8^2 = 128$ **(E)**.

3. **Problem:** What is the slope of a line perpendicular to the line through (-5,2) and (-1,0)?

   Solution: The slope of the original line is $\frac{0-2}{-1-(-5)} = -\frac{1}{2}$. The slope of a line perpendicular to this line will have a slope equal to the negative reciprocal of the slope of the original line. Therefore, the slope of the perpendicular line is $-\frac{1}{m} = -\frac{1}{-1/2} = 2$ **(D)**.

4. **Problem:** In convex hexagon $ABCDEF$, the average of angles A and D is $100°$. What is the average measure, in degrees, of angles B, C, E, and F.

   Solution: From our given information, we can determine that $A + B = 2 \cdot 100° = 200°$. Moreover, we know that the sum of the interior angles of an n-gon is $(n-2) \cdot 180°$, therefore, we know $A + B + C + D + E + F = 720°$.

   $$\begin{aligned} A + B + C + D + E + F &= 720\\ A + D &= 200\\ B + C + E + F &= 520 \end{aligned}$$

   Therefore, $\frac{B+C+E+F}{4} = 130$ **(D)**.

5. **Problem:** What is the sum of the digits of the largest palindrome less than 20152015?

Solution: If we keep the first four digits, the palindrome produced is 20155102, but this number is too large. The next smallest palindrome is 20144102, which satisfies the conditions. The sum of the digits is 14 **(E)**.

6. **Problem:** The pages of a book are numbered consecutively from 1 to 216. What is the sum of all the prime digits that appear in the page numbers? (Each digit is counted as many times as it appears.)

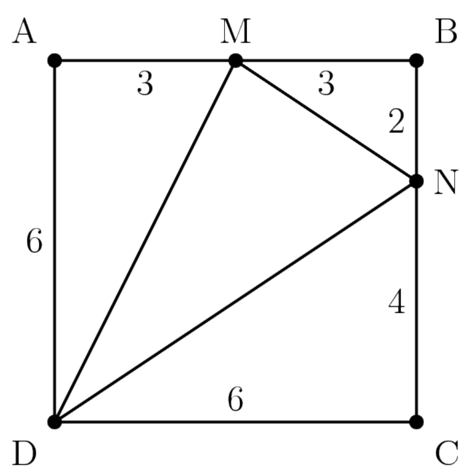
   Solution: We use casework to count in an organized manner. For each one-digit prime, we will find how many times it occurs from 0-99, from 100-199, and from 200-216. We will add these at the end.

   $2 \rightarrow 20+20+19=59$
   $3 \rightarrow 20+20+2=42$
   $5 \rightarrow 20+20+2=42$
   $7 \rightarrow 20+20+1=41$

   To find the sum, we multiply the prime with how many times it shows up, then find the sum of those products. $2 \cdot 59 + 3 \cdot 42 + 5 \cdot 42 + 7 \cdot 41 = 741$ **(B)**.

7. **Problem:** In square $ABCD$, let M be the midpoint of AB and N be the point on BC such that $CN = 2 \cdot BN$. Compute the ratio of the area of triangle DMN to the area of square $ABCD$.

   Solution: First, we must draw the diagram. To do so, we arbitrarily assign $AD = 6$.



   From this, we find that $[DMN] = 36 - 9 - 3 - 12 = 12$. Lastly, $\frac{12}{36} = \frac{1}{3}$ **(D)**.

8. **Problem:** Consider the circle $x^2 + y^2 = a^2$, where a is any positive number. Let ℓ be the line $y = b$, where b is a real number with $|b| < a$. The y-axis and ℓ divide the circle into 4 regions. Suppose the area of the upper right region is R_1 and going counter-clockwise, the other regions have areas R_2, R_3, and R_4. If $x = R_1 - R_4 + R_3 - R_2$, which of the following is true of x?

   Solution: We notice a symmetry around the y-axis. This means that $R_1 = R_2$ and $R_3 = R_4$. Therefore, $x = R_1 - R_4 + R_3 - R_2 = R_1 - R_2 + R_3 - R_4 = 0$ **(E)**.

9. **Problem:** Charlie is printing out all possible 5 digit license plates with letters and numbers, where the first and the last digit must be a vowel (A, E, I, O, U), and the middle three digits are digits from 0 to 9. If digits may not be repeated but vowels can, how many license plates will Charlie print in total?

   Solution: There are 5 possible letters for the first and last letters. For the numbers, there is 10 ways to choose the first number, 9 ways to choose the second number, and 8 ways to choose the last number. Therefore, by the fundamental counting principle, there are a total of $5 \cdot 5 \cdot 10 \cdot 9 \cdot 8 = 18000$ **(B)**.

10. **Problem:** A positive two-digit integer n is the sum of the product of its digits and the sum of its digits. Let P be the product of all possible values of n. What is the units digit of P.

    Solution: Let n be the number xy, where x and y are digits and $x \neq 0$. As such, n can be expressed as $10x + y$. Combining this with the original condition, we have:

    $$\begin{aligned} 10x + y &= xy + x + y \\ 9x &= xy \\ y &= 9 \end{aligned}$$

    Therefore, n is of the form $x9$, with no restrictions on x. This gives 9 possibilities: 19, 29, $\ldots$, 99. Multiplying these together will have the same units digit as 9^9. Observing the repeating pattern, we find that the units digit of 9^9 is 9 **(A)**.

11. **Problem:** How many positive integer cubes divide $15 \cdot 14 \cdots 1$?

    Solution: The number of times a prime p divides a number $n!$ can be found by computing the following:

    \# times p divides $n! = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \ldots$ (repeat until you get zero)

    From this formula, we can find the number of times each prime divides n:

    $2 \rightarrow 7 + 3 + 1 = 11$
    $3 \rightarrow 5 + 1 = 6$
    $5 \rightarrow 3$
    (and < 3 for the other primes)

    Because we need a perfect cube, our number may have 0, 3, 6, or 9 2's (4 values), 0, 3, or 6 3's (3 values), and 0 or 3 5's (2 values). From the fundamental counting theorem, we get 24 different numbers **(A)**.

12. **Problem:** What is the largest coefficient in the expansion of $(x + 4)^{10}$?

    Solution: The coefficients are in the form $4^k\binom{10}{k}$ for k in the range [0,10]. The coefficients are going to go up, then go down. We want to find the minimum k such that $\frac{4^{k+1}\binom{10}{k+1}}{4^k\binom{10}{k}} < 1 \rightarrow \frac{\binom{10}{k+1}}{\binom{10}{k}} < \frac{1}{4}$. We can simplify $\frac{\binom{10}{k+1}}{\binom{10}{k}} = \frac{10-k}{k+1}$. The minimum k such that $\frac{10-k}{k+1} < \frac{1}{4}$ is $k = 8$. Therefore, our answer is $4^8\binom{10}{2} = 2949120$ **(C)**.

13. **Problem:** If real numbers x and y, and k are such that $x^2 + y^2 = 4$ and $x^2 - 4y = k$, what is the smallest possible value of k?

Solution: To minimize k, we want to minimize $|x|$ and maximize y. This occurs when $x = 0$ and $y = 2$. $0^2 - 4 \cdot 2 = -8$ **(A)**.

14. **Problem:** James needs to dress up for his graduation speech. He needs to wear a pair of shoes, a pair of socks, a tie, a shirt, and pants. He has five of each of these five items, one red, one blue, one white, one black, and one green. He must pick one of each item, and he will wear any combination as long as it is not the case that all five items are of different colors. How many possible combinations of outfits can he wear?

    Solution: There are a total of 5^5 combinations if we did not have the condition that they cannot all be different colors. There are $5! = 120$ combinations with all 5 colors. $5^5 - 120 = 3005$ **(A)**.

15. **Problem:** What is the sum of the distinct positive composite divisors of 2016?

    Solution: $2016 = 2^5 \cdot 3^2 \cdot 7^1$. We can use the sum of divisors formula. Given that $n = p_1^{a_1} \cdot p_2^{a_2} \cdot p_3^{a_3} \ldots \cdot p_n^{a_n}$, then $\sigma(n) = (p_1^{a_1} + p_1^{a_1-1} + \ldots p_1 + 1)(p_2^{a_2} + p_2^{a_2-1} + \ldots p_2 + 1) \ldots (p_n^{a_n} + p_n^{a_n-1} + \ldots p_n + 1)$. $\sigma(2016) = 6552$, but we must take out the prime divisors. Since the non-composite divisors of 2016 are 1, 2, 3, and 7, we must subtract 13 from $\sigma(2016)$. Therefore, our answer is $6552 - 13 = 6539$ **(A)**.

16. **Problem:** Let a be the number of positive integer divisors of 2016. Compute the sum of the positive integer divisors of a.

    Solution: $2016 = 2^5 \cdot 3^2 \cdot 7^1$. We can use the divisor function to find the number of positive integer divisors. Given that $n = p_1^{a_1} \cdot p_2^{a_2} \cdot p_3^{a_3} \ldots \cdot p_n^{a_n}$, then $d(n) = (a_1+1)(a_2+1)(a_3+1) \ldots (a_n+1)$. Therefore, $d(2016) = 91$ **(A)**.

17. **Problem:** The orthocenter of a triangle is the point where the altitudes intersect. Triangle NAY is defined by points $N(0,0)$, $A(3,6)$, and $Y(5,0)$. If the triangle is rotated 90 degrees counter-clockwise around its orthoenter, what are the coordinates of the point A moves to?

    Solution: Because AY is parallel to the x-axis, we know that the altitude from A is $x = 3$. The slope of AY is $\frac{6-0}{3-5} = -3$, so the slope of the altitude from n is the negative reciprocal, or $\frac{1}{3}$. We want to find the intersection between $y = \frac{1}{3}x$ and $x = 3$. From this method, we find that the orthocenter is $(3,1)$. Rotating A 90° counter-clockwise around $(3,1)$ gives us $(-2,1)$ **(C)**.

18. **Problem:** For real numbers x, let $\{x\} = x - \lfloor x \rfloor$ (where $\lfloor x \rfloor$ denotes the greatest integer function). Compute the sum of all values of x that satisfy $\{x\} + 1 = \lfloor x^2 + 2x + 1 \rfloor$.

    Solution: The RHS has to be an integer, and it follows that the frational part of x has to be 0. Therefore, $(x+1)^2 = 1 \rightarrow |x+1| = 1 \rightarrow x = 0, -2$. The sum of these values of x is -2 **(E)**.