Sprint Test Round 11100 Solutions

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1 Solutions

1. **Problem:** If Bob flips a fair coin four times, what is the probability that he will get more heads than tails?

   Solution: There are $2^4 = 16$ outcomes for flipping 4 coins. More heads than tails means getting 3 or 4 heads. The number of ways to get both of these quantities is $\binom{4}{3} = 4$ and $\binom{4}{4} = 1$. So, there are a total of 5 favorable outcomes, giving a probability of $\frac{5}{16}$. **(D)**

2. **Problem:** If two workers can build a house in twenty days, how many days will it take sixteen workers to build six houses?

   Solution: Based on this information, it takes 40 worker-days to build a house. Thus, 6 houses require 240 worker-days. 240 worker-days divided by 16 workers leaves 15 days. **(A)**

3. **Problem:** A triangle has side lengths 5, 7, and x. What is the sum of all possible integer values of x?

   Solution: In a triangle with sides a, b, c, the inequality $a + b > c$ must be true. Therefore, the minimum value of x is 3, since $5 + x > 7$. Furthermore, the maximum value of x is 11, since $5 + 7 > x$. The sum of all integers 3 to 11 is $\frac{11 \cdot 12}{2} - 3 = 63$. **(C)**

4. **Problem:** Compute the area of the triangle formed by the x-axis, the y-axis, and the line $2x + 3y = 24$.

   Solution: The x-intercept is 12 and the y-intercept is 8, so the desired area is $\frac{8 \cdot 12}{2} = 48$. **(A)**

5. **Problem:** For how many integers x is $x^2 - 2011x + 2011$ negative?

   Solution: Given[1]

6. **Problem:** How many 3-digit palindromes are there?

   Solution: There are 9 possibilities for the 1st and 3rd digit (since they cannot be 0), and the middle digit can be any digit. $9 \cdot 10 = 90$. **(B)**

7. **Problem:** What is the sum of the absolute values of all integers b such that $x^2 + bx + 2011$ has integer roots?

[1] Provided in the official solutions.

Solution: 2011 is prime. Thus, its only factors are 2011 and 1 or -2011 and -1. Therefore, b can only be either 2012 or -2012. The sum of their absolute values is 4024. **(E)**

8. **Problem:** Alan, Ben, Casey, and Dan are sitting in a row of four chairs. If Ben and Casey are not sitting next to each other, in how any different arrangements could they four people be seated?

   Solution: There are $4! = 24$ ways to order the four people. Ben and Casey can sit together in 3 positions. Each position has 4 permutations, for $3 \cdot 4 = 12$ permutations where Ben and Casey are adjacent. This gives $24 - 12 = 12$ arrangements by complimentary counting. **(C)**

9. **Problem:** $2f(x) + 3f(5-x) = 7x$. Find $f(2)$.

   Solution: Given.

10. **Problem:** Two pencils, one eraser, and one pen cost 4 dollars. One pencil, three erasers, and two pens cost 6 dollars. Three pencils, one eraser, and four pens cost 9 dollars. What is the cost of six pencils, five erasers, and seven pens in dollars?

    Solution: Write equations for each statement. We will represent p as the number of pencils, e as the number of erasers, and n as the number of pens.

$$\begin{aligned} 2p + 3 + n &= 4 \\ p + 3e + 2n &= 6 \\ 3p + e + 4n &= 9 \end{aligned}$$

    Adding the above equations together, we find our desired value: $6p + 5e + 7n = 19$. **(D)**

11. **Problem:** What is the value of $\frac{\sqrt{8}-\sqrt{2}}{\sqrt{2}}$?

    Solution:

$$\frac{\sqrt{8}-\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}-\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}} = 1 \textbf{ (E)}$$

12. **Problem:** What is the are of the region in the xy-plane given by $|2x - 3y| \leq 12$ and $|2x + 3y| \leq 12$?

    Solution: Given.

13. **Problem:** The average of four different positive integers is 8. What is the largest possible value of any of these integers?

    Solution: Use the smallest possible integers are the first three integers: 1, 2, and 3. Therefore,

$$\begin{aligned} \frac{1+2+3+x}{4} &= 8 \\ x &= 26 \textbf{ (B)} \end{aligned}$$

14. **Problem:** What is the sum of the roots of the equation $(x^2 - 5x)^2 = 2(x^2 - 5x) + 24$?

    Solution: Given.

15. **Problem:** The graph of the function $y = b^x$ passes through the points (2,5) and (5,n). What is bn?

    Solution: Given.

16. **Problem:** What is the units digit of $200^2 - 199^2 + 198^2 - 197^2 + \ldots + 2^2 - 1^2$?

    Solution: This large expression can be simplified using the difference of squares. Since the sequence is consecutive, we find the following:

$$\begin{aligned} 200^2 - 199^2 + 198^2 - 197^2 + \ldots + 2^1 - 1^2 &= (200 + 199)(1) + (198 + 197)(1) + \ldots + (2 + 1)(1) \\ &= 200 + 199 + 198 + 197 + \ldots + 2 + 1 \\ &= \frac{200 \cdot 201}{2} \\ &= 20100 \end{aligned}$$

    Thus, the units digit is 0. **(A)**

17. **Problem:** How many digits are in the integer $17^{10,000}$?

    Solution: Given.

18. **Problem:** After an $x\%$ reduction, what increase does it take to restore the original price of an item?

    Solution: An $x\%$ reduction is equivalent to multiplying the original number by $\frac{100-x}{100}$. We must multiply this by $\frac{100}{100-x}$ to bring it back to its original price. Subtracting 1 gives us $\frac{x}{100-x}$, or the increase as a fraction. Multiplying this by 100 gives us the percent increase, $\frac{100x}{100-x}$. **(E)**

19. **Problem:** If $\sin(x + y) = 0.3$ and $\sin(x - y) = 0.5$, what is $\sin(x)\cos(y)$?

    Solution:

$$\begin{aligned} \sin(x + y) &= \sin(x)\cos(y) + \sin(y)\cos(x) \\ \sin(x - y) &= \sin(x)\cos(y) - \sin(y)\cos(x) \end{aligned}$$

    Let $\sin(x)\cos(y) = a$ and $\sin(y)\cos(x) = b$. We have $a + b = 0.3$ and $a - b = 0.5$. Adding these together and dividing by 2 tells us $a = \sin(x)\cos(y) = 0.4$. **(C)**

20. **Problem:** Two circles have the same center. A tangent from the inner circle is drawn and meets the outer circle at points A and B. If $AB = 12$, what is the difference in the areas of the circles, divided by π?

    Solution: Draw the radius from the center of the circle to AB. Call this length r. Draw the radius A from the center of the circle to B. Call this length R. We have $r^2 + 6^2 = R^2 \leftarrow R^2 - r^2 = 36$. Notice that the difference in the areas between the two circles is $\pi(R^2 - r^2) = 36\pi$. Dividing this my π gives us our answer, 36. **(C)**

21. **Problem:** Let T be a set. Let $P(T)$ be the set of all subsets of T. A subset S of $P(T)$ is "special" if $\emptyset \in S$, the intersection of any two sets in S is in S, and the union of any two sets in S is in S. Which of the following is not a special set?

(A) The set of all subsets of 1,2,3,4,5.
(B) The set of all subsets of 1,2,3,4,5 with even size (this includes $\emptyset$).
(C) The set $\emptyset$, 1,2, 3, 1,2,3
(D) The set of all subsets of the set of positive integers.
(E) Other

Solution: Check every set in S with every other to determine whether the intersection and union also exist in the set. For example, in choice A, since all subsets are represented, every union and intersection will also be a subset. However, in choice B, An intersection or union of 2 sets with even size can be of odd size when both subsets share 1 element. Thus, it is not special. **(B)**

22. **Problem:** How many integers between 1 and 1000 inclusive have exactly 27 positive divisors?

    Solution: To find the number of factors of an integer, add 1 to each of the exponents in its prime factorization and multiply the results. This can only be 27 if the integer is of the form ${p_1}^2 \cdot {p_2}^2 \cdot {p_3}^2$, ${p_1}^8 \cdot {p_2}^2$, or ${p_1}^{26}$. Checking these cases gives us the integer $2^2 \cdot 3^2 \cdot 5^2$ as our only integer. Therefore, there is only 1 desired integer. **(B)**

23. **Problem:** The sum of the length, width and height of a rectangular prism is 13. If the surface area is 48, what is the length of its space diagonal?

    Solution: Given.

24. **Problem:** The parabola $y = -x^2+4$ has roots A and B, with A to the left of B. The parabola is translated so that its vertex moves along the line $y = x + 4$, and B is still one of its roots. In which of the following intervals does the other root of the translated parabola lie in?

    Solution: From the equation, $y = -x^2 + 4$, we can find that B =(2,0). Since the vertex moves along the line $y = x + 4$, for every one unit the x-position changes, the y-position also changes by one. Therefore, the new equation satisfies $y = -(x - a)^2 + 4 + a$, for some a. We know this new equation also has a root at (2,0). Therefore,

$$
\begin{aligned}
y &= -x^2 + 2ax - a^2 + 4 + a \\
0 &= -4 + 4a - a^2 + 4 + a \\
0 &= 5a - a^2 \\
a &= 0 \text{ or } a = 5
\end{aligned}
$$

    Since $a = 0$ is our original case, we use $a = 5$. As such, we get the equation $y = -(x-5)^2+9 = -x^2 + 10x - 16$. This has roots at (8,0) and (2,0). Therefore, the right root is in the interval [8,9). **(C)**

25. Rest of the problems have solutions[2].

[2]Provided in the official solutions.